JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **34**, No. 1, February 2021
http://dx.doi.org/10.14403/jcms.2021.34.1.93

VANISHING OF GROMOV-WITTEN INVARIANTS OF PRODUCT OF $\mathbb{P}^1$

Hyenho Lho

Abstract. We study the Gromov-Witten invariants of $\underbrace{\mathbb{P}^1 \times \mathbb{P}^1 \times ... \times \mathbb{P}^1}_{n}$. Our main result is a proof of some vanishing conditions on the Gromov-Witten invariants of $\mathbb{P}^1 \times \mathbb{P}^1 \times ... \times \mathbb{P}^1$ in all genera.

1. Introduction

1.1. Gromov-Witten invariants

Denote by $\mathbb{P}[n]$ the product of n projective lines:

$$\mathbb{P}[n] := \underbrace{\mathbb{P}^1 \times \mathbb{P}^1 \times ... \times \mathbb{P}^1}_{n}.$$

Denote by

$$\overline{M}_{g,m}(\mathbb{P}[n], \mathsf{d})$$

the moduli space of stable maps to $\mathbb{P}[n]$ with degree $\mathsf{d} = (d_1, d_2, ..., d_n)$. It has the canonical virtual fundamental class $[\overline{M}_{g,m}(\mathbb{P}[n], \mathsf{d})]^{vir}$ of dimension $(1-g)(n-3) + m + 2 \cdot \sum_i d_i$.
The Gromov-Witten invariants without insertions are defined by

$$GW_{g,0}(\mathbb{P}[n], \mathsf{d}) := \int_{[\overline{M}_{g,0}(\mathbb{P}[n],\mathsf{d})]^{vir}} 1.$$

The Gromov-Witten invariants have been studied more than 20 years, see [6],[7] for an introduction to the subject. By dimensional reason, $GW_{g,0}(\mathbb{P}[n], \mathsf{d}) = 0$ unless $(g-1)(n-3) = 2\sum_k d_k$. Even though there

Received January 07, 2021; Accepted February 08, 2021.
2010 Mathematics Subject Classification: 14D23.
Key words and phrases: Gromov-Witten invariant, Stable map.

exist many way to calculate the Gromov-Witten invariants of $\mathbb{P}[n]$, they are not efficient to get some universal results for higher genus.

In this paper, we prove the following vanishing conditions on the Gromov-Witten invariants of $\mathbb{P}[n]$ without insertions from a combinatorial observation obtained via torus localization.

THEOREM 1. *If g is even and $n > 3g-3$,*

$$GW_{g,0}(\mathbb{P}[n], \mathsf{d}) = 0.$$

Theorem 1 can be extended to Gromov-Witten invariants with insertions. To keep the notations simple we state the general theorem in Section 4.

1.2. Quasimap invariants and wall-crossing formula

Let $\mathbf{G} := (\mathbb{C}^*)^n$ act on $(\mathbb{C}^2)^n$ by the standard diagonal action componentwisly on each component $\mathbb{C}^2$ so that its associated GIT quotient is

$$(\mathbb{C}^2)^n /\!\!/ \mathbf{G} = \mathbb{P}[n].$$

With this set-up, Ciocan-Fontanine and Kim defined quasimap moduli space

$$Q_{g,m}(\mathbb{P}[n], \mathsf{d})$$

with the canonical virtual fundamental class $[Q_{g,k}(\mathbb{P}[n], \mathsf{d})]^{vir}$. See [1], [5], [15] for an introduction. We define quasimap invariants of $\mathbb{P}[n]$ without insertions by

$$QI_{g,0}(\mathbb{P}[n], \mathsf{d}) := \int_{[Q_{g,0}(\mathbb{P}[n],\mathsf{d})]^{vir}} 1.$$

In [3], authors studied the relationship between quasimap invariants and Gromov-Witten invariants. By applying the general theorem to $\mathbb{P}[n]$, We have

PROPOSITION 2. ([3])

$$GW_{g,0}(\mathbb{P}[n], \mathsf{d}) = QI_{g,0}(\mathbb{P}[n], \mathsf{d})$$

i.e. quasimap invariants and Gromov-Witten invariants are same for $\mathbb{P}[n]$. In section 3, we prove the following theorem which is equivalent to Theorem 1 by Proposition 2.

THEOREM 3. *If g is even and $n > 3g-3$,*

$$QI_{g,0}(\mathbb{P}[n], \mathsf{d}) = 0.$$

1.3. Acknowledgments

I am grateful to I. Ciocan-Fontanine, B. Kim, R. Pandharipande and J. Shen for discussions over the years about the moduli space of quasimaps and the Gromov-Witten invariants. This work was supported by research fund of Chungnam National University.

2. Preliminaries

2.1. T-equivariant theory

Let $\mathsf{T} := (\mathbb{C}^*)^{2n}$ act on $\mathbb{P}[n]$ standardly on each component. Denote by

$$\lambda = (\lambda_1, \overline{\lambda}_1, \lambda_2, \overline{\lambda}_2, ..., \lambda_n, \overline{\lambda}_n)$$

T-equivariant parameters. We will use following specializations throughout the paper;

$$\lambda_k + \overline{\lambda}_k = 0\,, \text{ for } 1 \leq k \leq n.$$

First we set the notation for the cohomology basis and its dual basis. Let $\{p_i\}$ be the set of T-fixed points of $\mathbb{P}[n]$ and let ϕ_i be the basis of $H^*_{\mathsf{T}}(\mathbb{P}[n])$ defined by satisfying followings:

$$\phi_i|_{p_j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j\,. \end{cases} \tag{1}$$

Let ϕ^i be the dual basis with respect to the T-equivariant Poincaré pairing, i.e.,

$$\int_Y \phi_i \phi^j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j\,, \end{cases}$$

For each T-fixed point $p_i \in \mathbb{P}[n]$, Let

$$e_i = e(T_{p_i}(\mathbb{P}[n]))$$

be the equivariant Euler class of the tangent space of $\mathbb{P}[n]$ at p_i.

The action of T on $\mathbb{P}[n]$ lifts to $Q_{g,0}(\mathbb{P}[n], \mathsf{d})$. The localization formula of [11] applied to the virtual fundamental class $[Q_{g,0}(\mathbb{P}[n], \mathsf{d})]^{\mathrm{vir}}$ will play a fundamental role in our paper. The T-fixed loci are represented in terms of dual graphs, and the contributions of the T-fixed loci are given by tautological classes. The formulas are standard, see [12],[15].

2.2. Genus zero invariants

In this section, we review the genus zero theory. Integrating along the virtual fundamental class

$$[Q_{0,k}(\mathbb{P}[n],\mathsf{d})]^{\mathrm{vir}}$$

we define correlators $\langle ... \rangle^{0+}_{0,k,\mathsf{d}}$ as follows. For $\gamma_i \in H^*_{\mathsf{T}}(\mathbb{P}[n]) \otimes \mathbb{Q}(\lambda)$,

$$\langle \gamma_1\psi^{a_1}, ..., \gamma_k\psi^{a_k}\rangle^{0+}_{0,k,\mathsf{d}} = \int_{[Q_{0,k}(\mathbb{P}[n],\mathsf{d})]^{\mathrm{vir}}} \prod_i ev_i^*(\gamma_i)\psi_i^{a_i},$$

where ψ_i is the psi-class associated to the i-th marking and ev_i is the i-th evaluation map.

Let $Q_{g,k}(\mathbb{P}[n],\mathsf{d})^{\mathsf{T},p_i}$ be the fixed loci of $Q_{g,k}(\mathbb{P}[n],\mathsf{d})$ whose elements have domain components only over p_i. Integrating along the localized cycle class

$$\frac{[Q_{0,k}(\mathsf{d})^{\mathsf{T},p_i}]^{\mathrm{vir}}}{\mathrm{e}^{\mathsf{T}}(N^{\mathrm{vir}}_{Q_{0,k}(\mathbb{P}[n],\mathsf{d})^{\mathsf{T},p_i}/Q_{0,k}(\mathbb{P}[n],\mathsf{d})})}$$

we also define local correlators $\langle ... \rangle^{0+,p_i}_{0,k,\beta}$ and $\langle\langle ... \rangle\rangle^{0+,p_i}_{0,k,\beta}$ as follows:

$$\langle \gamma_1\psi^{a_1}, ..., \gamma_k\psi^{a_k}\rangle^{0+,p_i}_{0,k,\mathsf{d}} = \int_{\frac{[Q_{0,k}(\mathbb{P}[n],\mathsf{d})^{\mathsf{T},p_i}]^{\mathrm{vir}}}{\mathrm{e}^{\mathsf{T}}(N^{\mathrm{vir}}_{Q_{0,k}(\mathbb{P}[n],\mathsf{d})^{\mathsf{T},p_i}/Q_{0,k}(\mathbb{P}[n],\mathsf{d})})}} \prod_i ev_i^*(\gamma_i)\psi_i^{a_i} \quad ;$$

$$\langle\langle \gamma_1\psi^{a_1}, ..., \gamma_k\psi^{a_k}\rangle\rangle^{0+,p_i}_{0,k}$$
$$= \sum_{m,\mathsf{d}} \frac{\mathsf{q}^{\mathsf{d}}}{m!}\langle \gamma_1\psi^{a_1}, ..., \gamma_k\psi^{a_k}, \mathsf{t}, ..., \mathsf{t}\rangle^{0+,p_i}_{0,k+m,\mathsf{d}},$$

where $\mathsf{q} = (q_1, \ldots, q_n)$ are formal Novikov variables and $\mathsf{t} = \sum_i t_i H_i \in H^*_{\mathsf{T}}(\mathbb{P}[n])$ where H_i is the pull back of hyperplane class in i-th $\mathbb{P}^1$.

2.3. Insertions of 0+ weighted marking

For the explicit calculations of various genus 0 invariants, we need to introduce the notion of 0+ weighted marking. We briefly recall the definitions from [4].

Denote by

$$Q^{0+,0+}_{g,k|m}(\mathbb{P}[n],\mathsf{d})$$

the moduli space of genus g (resp. genus zero), degree d stable quasimaps to $\mathbb{P}[n]$ with ordinary k pointed markings and infinitesimally weighted m pointed markings, see [4] for more explanations.
Denote by

$$Q^{0+,0+}_{g,k|m}(\mathbb{P}[n],\mathsf{d})^{\mathsf{T},p_i}$$

the T-fixed part of $Q^{0+,0+}_{g,k|m}(\mathbb{P}[n],\mathsf{d})$, whose domain components of universal curves are only over p_i.

Let $\tilde{H}_i \in H^*([(\mathbb{C}^2)^n/\mathbf{G}])$ be a lift of $H_i \in H^*(\mathbb{P}[n])$, i.e. , $\tilde{H}_i|_{\mathbb{P}[n]} = H_i$. For $\gamma_i \in H^*_{\mathsf{T}}(\mathbb{P}[n]) \otimes \mathbb{Q}(\lambda)$, $\mathsf{t} = \sum_i t_i \tilde{H}_i, \delta_j \in H^*_{\mathsf{T}}([(\mathbb{C}^2)^n/\mathbf{G}], \mathbb{Q})$, denote

$$\begin{aligned}
&\langle \gamma_1\psi^{a_1}, ..., \gamma_k\psi^{a_k}; \delta_1, ..., \delta_m\rangle^{0+,0+}_{0,k|m,\mathsf{d}} \\
&= \int_{[Q^{0+,0+}_{0,k|m}(\mathbb{P}[n],\mathsf{d})^{\mathsf{T}}]^{\mathrm{vir}}} \prod_i ev_i^*(\gamma_i)\psi_i^{a_i} \prod_j \hat{ev}_j^*(\delta_j) \quad ; \\
&\langle\langle \gamma_1\psi^{a_1}, ..., \gamma_k\psi^{a_k}\rangle\rangle^{0+,0+}_{0,k} \\
&= \sum_{m,\beta} \frac{\mathsf{q}^{\mathsf{d}}}{m!} \langle \gamma_1\psi^{a_1}, ..., \gamma_k\psi^{a_k}; \mathsf{t}, ..., \mathsf{t}\rangle^{0+,0+}_{0,k|m,\mathsf{d}} \quad ; \\
&\langle \gamma_1\psi^{a_1}, ..., \gamma_k\psi^{a_k}; \delta_1, ..., \delta_m\rangle^{0+,0+,p_i}_{0,k|m,\mathsf{d}} \\
&= \int_{\frac{[Q^{0+,0+}_{0,k|m}(\mathbb{P}[n],\mathsf{d})^{\mathsf{T},p_i}]^{\mathrm{vir}}}{\mathrm{e}^{\mathsf{T}}(N^{\mathrm{vir}}_{Q^{0+,0+}_{0,k|m}(\mathbb{P}[n],\mathsf{d})^{\mathsf{T},p_i}/Q^{0+,0+}_{0,k|m}(\mathbb{P}[n],\mathsf{d})})}} \prod_i ev_i^*(\gamma_i)\psi_i^{a_i} \prod_j \hat{ev}_j^*(\delta_j) \quad ; \\
&\langle\langle \gamma_1\psi^{a_1}, ..., \gamma_k\psi^{a_k}\rangle\rangle^{0+,0+,p_i}_{0,k} \\
&= \sum_{m,\mathsf{d}} \frac{\mathsf{q}^{\mathsf{d}}}{m!} \langle \gamma_1\psi^{a_1}, ..., \gamma_k\psi^{a_k}; \mathsf{t}, ..., \mathsf{t}\rangle^{0+,0+,p_i}_{0,k|m,\mathsf{d}} \quad ,
\end{aligned}$$

where $\hat{ev}_j$ is the evaluation map to $[(\mathbb{C}^2)^n/\mathbf{G}]$ at the j-th infinitesimally weighted marking.

Let $\{\phi_i\}$ be the equivariant basis of $H^*_{\mathsf{T}}(\mathbb{P}[n])$ satisfying (1). Let us define $\mathbb{U}_i$, $\mathbb{S}$, $\mathbb{V}_{ij}$ by

$$\mathbb{U}^{\mathbb{P}[n]}_{p_i} := \langle\langle 1, 1\rangle\rangle^{0+,0+,i}_{0,2} \in \mathsf{R}[[\mathsf{q},\mathsf{t}]] \quad ;$$

$$\mathbb{S}^{\mathbb{P}[n]}(\gamma) := \sum_{i,\mathsf{d}} \phi^i \mathsf{q}^{\mathsf{d}} \langle\langle \frac{\phi_i}{z-\psi}, \gamma\rangle\rangle^{0+,0+}_{0,2,\mathsf{d}} \in \mathsf{R}[[z^{-1},\mathsf{q},\mathsf{t}]] \quad ;$$

$$\mathbb{V}^{\mathbb{P}[n]}_{ij}(x,y) := \sum_{\mathsf{d}} \mathsf{q}^{\mathsf{d}} \langle\langle \frac{\phi_i}{x-\psi}, \frac{\phi_j}{y-\psi}\rangle\rangle^{0+,0+}_{0,2,\mathsf{d}} \in \mathsf{R}[[x^{-1},y^{-1},\mathsf{q},\mathsf{t}]] \quad ;$$

We recall the following WDVV equation from [3],

$$e_i \mathbb{V}_{ij}(x,y) e_j = \frac{\sum_k \mathbb{S}(\phi_k)|_{p_i,z=x} \mathbb{S}(\phi^k)|_{p_j,z=y}}{x+y} \tag{2}$$

To study the properties of $\mathbb{S}$ and $\mathbb{V}_{ij}$, we need to recall infinitesimal I-function defined in [4]. Denote by $\mathbb{I}$ the infinitesimal I-function $\mathbb{J}^{0+,0+}$ in [4]. Note that $\mathbb{S}$ coincides with the definition of infinitesimal S-operator $\mathbb{S}^{0+,0+}$ in [4]. From (5.1.3) in [4], we know explicit form of $\mathbb{I}$ of $\mathbb{P}[n]$

$$\begin{aligned} \mathbb{I}^{\mathbb{P}[n]} = \prod_{i=1}^{n} e^{t_i(H_i+d_i z)/z} \frac{q_i^{d_i}}{\prod_{k=1}^{d_i}(H_i-\lambda_i+kz)(H_i-\overline{\lambda}_i+kz)} \\ \in \mathsf{R}[[z,z^{-1},\mathsf{q},\mathsf{t}]](\lambda). \end{aligned} \tag{3}$$

To find the explicit forms of $\mathbb{S}$-operators, we recall following proposition from [12].

Proposition 4. (Proposition 2.4 in [12]) There are unique coefficients $a_i(z,\mathsf{q}) \in \mathbb{Q}(\lambda)[z][[\mathsf{q}]]$ making

$$\sum_i a_i(z,\mathsf{q}) \partial_{\phi_i} \mathbb{I}^{\mathbb{P}[n]} = \gamma + O(1/z).$$

Furthermore LHS coincides with $\mathbb{S}^{\mathbb{P}[n]}(\gamma)$.

2.4. Ordered graphs.

Let the genus g and the number of markings m for the moduli space be in the stable range

$$2g-2+m>0.$$

We can organize the T-fixed loci of $Q_{g,m}(\mathbb{P}[n],\mathsf{d})$ according to decorated graphs. A *decorated graph* Γ consists of the data $(\mathsf{V},\mathsf{E},\mathsf{N},\mathsf{g})$ where

(i) V is the vertex set,
(ii) E is the edge set (including possible self-edge),
(iii) $\mathsf{N} : \{1, 2, \ldots, m\} \to \mathsf{V}$ is the marking assignment,
(iv) $\mathsf{g} : \mathsf{V} \to \mathbb{Z}_{\geq 0}$ is a genus assignment satisfying

$$g = \sum_{v \in \mathsf{V}} \mathsf{g}(v) + h^1(\Gamma).$$

Let $\mathsf{G}_{g,m}$ be the set of decorated graph as defined as above. The *flags* of Γ are the half-edges[1]. Let F be the set of flags. For each $\Gamma \in \mathsf{G}_{g,m}$, we choose an ordering

$$\nu_\Gamma : \mathsf{V} \to \{1, 2, \ldots, |\mathsf{V}|\}. \tag{4}$$

Let $\mathsf{G}_{g,m}^{\mathrm{ord}}$ be the set of decorated graph with fixed choice of orderings on vertices. We will sometimes identify V with $\{1, 2, \ldots, |\mathsf{V}|\}$ by (4).

2.5. Universal ring

Denote by

$$\mathsf{R}$$

the ring generated by

$$\lambda_{1,j}^{\pm 1}, \lambda_{2,j}^{\pm 1}, \ldots, \lambda_{n,j}^{\pm 1}, \ j \in \mathbb{N}$$

with relations

$$\lambda_{k,j}^2 = 1, \text{ for, } 1 \leq k \leq n, \ j \in \mathbb{N}. \tag{5}$$

- A monomial in R is called *canonical* if degree of $\lambda_{i,j}$ are 1 or 0 for all $1 \leq i \leq n$, $j \in \mathbb{N}$.
- *The length* of canonical monomial is the number of $\lambda_{i,j}$ whose degree is 1.
- For a monomial $m \in \mathsf{R}$, *the length* of m is the length of unique canonical monomial equal to m in R.

The ring R will play a fundamental role in the proof of the main theorem.

Definition 5. *For monomial $m \in \mathsf{R}$, we say m has type $(a, b) \in \mathbb{N}^2$ if m has following form;*

$$m = \prod_j \prod_i \lambda_{i,j}^{a_{i,j}} f$$

[1] Flags are either half-edges or markings.

where $\sum_j a_{i,j} = a$ *and* f *is a monomial whose length is less than or equal to* b.

DEFINITION 6. *For a polynomial* $f \in \mathsf{R}$, *we say* f *has type* (a, b) *if* f *is a sum of monomials of type* (a, b).

From the above definition, wee can easily check the following two lemmas.

LEMMA 7. *If* m_1 *has type* (a_1, b_1) *and* m_2 *has type* (a_2, b_2), *then* $m_1 m_2$ *has type* $(a_1 + a_2, b_1 + b_2)$.

LEMMA 8. *If* $f \in \mathsf{R}$ *has type* (a, b) *with odd* a *and* $n > b$,

$$\sum_{\lambda_{i,j}=\pm 1} f = 0.$$

The notation $\sum_{\lambda_{i,j}=\pm 1} f$ in Lemma 8 denotes the sum of evaluations of $f \in \mathsf{R}$ with all possible choices of $\lambda_{i,j} = 1 \,\text{or} - 1$.

2.6. Universal polynomial

We review here the definition of universal polynomial P from [14]. The universal polynomial play the essential role in calculating higher genus invariants from genus 0 invariants, see [9], [13], [14] for more explanations.

Let $t_0, t_1, t_2, \ldots$ be formal variables. The series

$$T(c) = t_0 + t_1 c + t_2 c^2 + \ldots$$

in the additional variable c plays a basic role. The variable c will later be replaced by the first Chern class ψ_i of a cotangent line over $\overline{M}_{g,n}$,

$$T(\psi_i) = t_0 + t_1 \psi_i + t_2 \psi_i^2 + \ldots\,,$$

with the index i depending on the position of the series T in the correlator.

Let $2g - 2 + n > 0$. For $a_i \in \mathbb{Z}_{\geq 0}$ and $\gamma \in H^*(\overline{M}_{g,n})$, define the correlator

$$\langle\langle \psi^{a_1}, \ldots, \psi^{a_n} \,|\, \gamma \rangle\rangle_{g,n} \;=\; \sum_{k \geq 0} \frac{1}{k!} \int_{\overline{M}_{g,n+k}} \gamma\, \psi_1^{a_1} \cdots \psi_n^{a_n} \prod_{i=1}^{k} T(\psi_{n+i})\,.$$

We consider $\mathbb{C}(t_1)[t_2, t_3, ...]$ as $\mathbb{Z}$-graded ring over $\mathbb{C}(t_1)$ with

$$\deg(t_i) = i - 1 \quad \text{for } i \geq 2\,.$$

Define a subring of homogeneous elements by

$$\mathbb{C}\left[\frac{1}{1-t_1}\right][t_2, t_3, \ldots]_{\mathrm{Hom}} \subset \mathbb{C}(t_1)[t_2, t_3, \ldots] .$$

We easily see

$$\langle\langle \psi^{a_1}, \ldots, \psi^{a_n} \mid \gamma \rangle\rangle_{g,n} |_{t_0=0} \ \in \ \mathbb{C}\left[\frac{1}{1-t_1}\right][t_2, t_3, \ldots]_{\mathrm{Hom}} .$$

Using the leading terms (of lowest degree in $\frac{1}{(1-t_1)}$), we obtain the following result.

Lemma 9. *The set of genus 0 correlators*

$$\Big\{ \langle\langle 1, \ldots, 1 \rangle\rangle_{0,n} |_{t_0=0} \Big\}_{n \geq 4}$$

freely generate the ring $\mathbb{C}(t_1)[t_2, t_3, \ldots]$ over $\mathbb{C}(t_1)$.

By Lemma 9, we can find a unique representation of $\langle\langle \psi^{a_1}, \ldots, \psi^{a_n} \rangle\rangle_{g,n}|_{t_0=0}$ in the variables

$$\Big\{ \langle\langle 1, \ldots, 1 \rangle\rangle_{0,n}|_{t_0=0} \Big\}_{n \geq 3} . \tag{6}$$

The $n = 3$ correlator is included in the set (6) to capture the variable t_1. For example, in $g = 1$,

$$\begin{aligned}
\langle\langle 1, 1 \rangle\rangle_{1,2}|_{t_0=0} &= \frac{1}{24}\left(\frac{\langle\langle 1,1,1,1,1 \rangle\rangle_{0,5}|_{t_0=0}}{\langle 1,1,1 \rangle\rangle_{0,3}|_{t_0=0}} - \frac{\langle\langle 1,1,1,1 \rangle\rangle_{0,4}^2|_{t_0=0}}{\langle\langle 1,1,1 \rangle\rangle_{0,3}^2|_{t_0=0}}\right) , \\
\langle\langle 1 \rangle\rangle_{1,1}|_{t_0=0} &= \frac{1}{24}\frac{\langle\langle 1,1,1,1 \rangle\rangle_{0,4}|_{t_0=0}}{\langle\langle 1,1,1 \rangle\rangle_{0,3}|_{t_0=0}}
\end{aligned}$$

A more complicated example in $g = 2$ is

$$\begin{aligned}
\langle\langle \rangle\rangle_{2,0}|_{t_0=0} = & \ \frac{1}{1152}\frac{\langle\langle 1,1,1,1,1,1 \rangle\rangle_{0,6}|_{t_0=0}}{\langle\langle 1,1,1 \rangle\rangle_{0,3}|_{t_0=0}^2} \\
& -\frac{7}{1920}\frac{\langle\langle 1,1,1,1,1 \rangle\rangle_{0,5}|_{t_0=0}\langle\langle 1,1,1,1 \rangle\rangle_{0,4}|_{t_0=0}}{\langle\langle 1,1,1 \rangle\rangle_{0,3}|_{t_0=0}^3} \\
& +\frac{1}{360}\frac{\langle\langle 1,1,1,1 \rangle\rangle_{0,4}|_{t_0=0}^3}{\langle\langle 1,1,1 \rangle\rangle_{0,3}|_{t_0=0}^4} .
\end{aligned}$$

Definition 10. *For $\gamma \in H^*(\overline{M}_{g,k})$, let*

$$\mathsf{P}_{g,n}^{a_1,\ldots,a_n,\gamma}(s_0, s_1, s_2, \ldots) \in \mathbb{Q}(s_0, s_1, ..)$$

be the unique rational function satisfying the condition

$$\langle\langle \psi^{a_1}, \ldots, \psi^{a_n} \mid \gamma \rangle\rangle_{g,n}|_{t_0=0} = \mathsf{P}_{g,n}^{a_1,a_2,\ldots,a_n,\gamma}|_{s_i=\langle\langle 1,\ldots,1 \rangle\rangle_{0,i+3}|_{t_0=0}} .$$

We use the following notation.

$$\mathsf{P}\left[\psi_1^{k_1},\ldots,\psi_n^{k_n}\,\middle|\,\mathsf{H}_h^{p_i}\right]_{h,n}^{0+,p_i} :=$$
$$\mathsf{P}_{h,1}^{k_1,\ldots,k_n,\mathsf{H}_h^{p_i}}\big(\langle\langle 1,1,1\rangle\rangle_{0,3}^{0+,p_i},\langle\langle 1,1,1,1\rangle\rangle_{0,4}^{0+,p_i},\ldots\big).$$

3. Higher genus series on $\mathbb{P}[n]$

We review here the standard method first used by Givental to express genus g descendent correlators in terms of genus 0 data. See [10],[13],[12] for more explanations.

Definition 11. *Let us define equivariant genus g generating function $F^g(\mathsf{q})\in\mathbb{C}[[\mathsf{q}]](\lambda)$ of $\mathbb{P}[n]$ as following;*

$$F^g(\mathsf{q}) := \sum_{d_1,d_2,..,d_n\geq 0} q_1^{d_1}q_2^{d_2}...q_n^{d_n}\int_{[Q_{g,0}(\mathbb{P}[n],\mathsf{d})]^{\mathrm{vir}}} 1$$
$$= \sum_{\mathsf{d}\geq 0}\mathsf{q}^{\mathsf{d}}\int_{[Q_{g,0}(\mathbb{P}[n],\mathsf{d})]^{\mathrm{vir}}} 1 \in \mathbb{C}[[\mathsf{q}]](\lambda).$$

We apply the localization strategy introduced first by Givental for Gromov-Witten theory to the quasimap invariats of $\mathbb{P}[n]$. We write the localization formula as

$$F^g|_{\lambda_i=1,\bar{\lambda}_i=-1} = \sum_{\lambda_{i,j}=\pm 1}\sum_{\Gamma\in\mathsf{G}_{g,0}^{\mathrm{ord}}}\mathrm{Cont}_\Gamma$$

where Cont_Γ is the contribution to F^g of the T-fixed loci associated Γ. We have the following formula for Cont_Γ.

$$\mathrm{Cont}_\Gamma = \frac{1}{\mathrm{Aut}(\Gamma)}\sum_{\mathsf{A}\in\mathbb{Z}_{\geq 1}^{F(\Gamma)}}\prod_v\prod_e \mathrm{Cont}_\Gamma^{\mathsf{A}}(v)\mathrm{Cont}_\Gamma^{\mathsf{A}}(e).$$

We need explanations for the $\mathrm{Cont}_\Gamma^{\mathsf{A}}(v)$ and $\mathrm{Cont}_\Gamma^{\mathsf{A}}(e)$. Let $v\in V(\Gamma)$ be a vertex of genus $\mathsf{g}(v)$.

$$\mathrm{Cont}_\Gamma^{\mathsf{A}}(v) = \mathsf{P}[\psi^{a_1-1},\psi^{a_2-1},...,\psi^{a_k-1}|\mathsf{H}_{\mathsf{g}(v)}^{v}]_{\mathsf{g}(v),k}^{0+,v}$$

where,

- $\mathsf{H}^v_{\mathsf{g}(v)} = \prod_{k=1}^n \frac{\prod_{j=1}^{\mathsf{g}(v)}((\lambda_{k,v}-\overline{\lambda}_{k,v})-c_j)}{(\lambda_{k,v}-\overline{\lambda}_{k,v})}$
- $k = val(v)$ is valency at v and $(a_1, a_2, ..., a_k)$ are component of $\mathsf{A} \in \mathbb{Z}_{\geq 1}^{F(\Gamma)}$ corresponding to flag of v.
- $\lambda_{k,v}$ is the weight of k-th $\mathbb{C}^*$ at $\nu(v)$.
- $\{c_j | 1 \leq j \leq \mathsf{g}(v)\}$ are chern roots of Hodge bundle on $\overline{M}_{\mathsf{g}(v),k}$.

Let $e \in E(\Gamma)$ be an edge between v_i and v_j.

$$\mathrm{Cont}^{\mathsf{A}}_{\Gamma}(e) = \left[e^{-\frac{\overline{\mathbb{U}}_i^{\mathbb{P}[n]}}{x} - \frac{\overline{\mathbb{U}}_j^{\mathbb{P}[n]}}{y}} e_i \overline{\mathbb{V}}_{ij}^{\mathbb{P}[n]}(x,y) e_j \right]_{x^{a_i-1}y^{a_j-1}}$$

where (a_i, a_j) are component of $\mathsf{A} \in \mathbb{Z}_{\geq 1}^{F(\Gamma)}$ associated to flags (v_i, e) and (v_j, e). The notation $[\ldots]_{x^{a_1-1}y^{a_2-1}}$ above denotes the coefficient of $x^{a_1-1}y^{a_2-1}$ in the series expansion of the argument. The bar above $\mathbb{U}_i$ and $\mathbb{V}_{ij}$ means evaluation $\mathsf{t} = 0$.

Remark 12. By (4), $\mathrm{Cont}^{\mathsf{A}}_{\Gamma}(v)$ and $\mathrm{Cont}^{\mathsf{A}}_{\Gamma}(v)$ can be considered as elements in $\mathsf{R}[[\mathsf{q}, x^{-1}, y^{-1}]]$.

Finally we obtain the following result.

Proposition 13. *For $\mathbb{P}[n]$, We have*

$$F^g|_{\lambda_i=1,\overline{\lambda}_i=-1} = \sum_{\lambda_{i,j}=\pm 1} \sum_{\Gamma \in \mathsf{G}^{\mathrm{ord}}_{g,0}} \frac{1}{\mathrm{Aut}(\Gamma)} \sum_{\mathsf{A} \in \mathbb{Z}_{\geq 1}^{F(\Gamma)}} \prod_v \prod_e \mathrm{Cont}^{\mathsf{A}}_{\Gamma}(\mathrm{v}) \mathrm{Cont}^{\mathsf{A}}_{\Gamma}(\mathrm{e}),$$

where $Cont^{\mathsf{A}}_{\Gamma}(v)$,$Cont^{\mathsf{A}}_{\Gamma}(e)$ are defined as above.

4. Proof of main theorem

4.1. Overview

Using (3) and Prop 4, we can explicitly calculate the $\mathbb{S}$-operator of $\mathbb{P}[n]$.

Proposition 14. *For $\{i_1, i_2, ..., i_n\} = \{0, 1\}$,*

$$\mathbb{S}^{\mathbb{P}[n]}(H_1^{i_1}, H_2^{i_2}, ..., H_n^{i_n}) = (z\frac{d}{dt_1})^{i_1}(z\frac{d}{dt_2})^{i_2}...(z\frac{d}{dt_n})^{i_n}\mathbb{I}^{\mathbb{P}[n]}.$$

If we restrict $\mathbb{S}$-operator to the fixed points, it admits Birkhoff factorizations, see [2].

Definition 15. *For fixed point $p \in (\mathbb{P}[n])^T$, let us define $\mathbb{R}_{p,k}^{i_1,i_2,..,i_n}$ by following;*

$$\mathbb{S}^{\mathbb{P}[n]}(H_1^{i_1}H_2^{i_2}...H_n^{i_n})|_p =$$
$$e^{\frac{\mathbb{U}_p^{\mathbb{P}[n]}}{z}}\lambda_{1,p}^{\delta_{1i_1}}\lambda_{2,p}^{\delta_{1i_2}}...\lambda_{n,p}^{\delta_{1i_n}}(\sum_k \mathbb{R}_{p,k}^{i_1,i_2,..,i_n}z^k) \in \mathsf{R}[[z,z^{-1},\mathsf{q},\mathsf{t}]](\lambda).$$

Here $\lambda_{k,p}$ is the weight of k-th $\mathbb{C}^*$ at p. We consider $\lambda_{k,p}$ as an element of R by identifying $\lambda_{k,p}$ with $\lambda_{k,1}$.

Proposition 16. $\mathbb{R}_{p,k}^{i_1,i_2,..,i_n}$ *has type $(0,k)$.*

Proof. Applying Prop 14 to (3), the proof follows from following lemma. □

Lemma 17. *We have following assymtotic expansion of local $\mathbb{S}$-operators of $\mathbb{P}^1$ for fixed point $p \in (\mathbb{P}^1)^T$.*

$$\mathbb{S}^{\mathbb{P}^1}(1)|_p = e^{\frac{\mathbb{U}_p^{\mathbb{P}^1}}{z}}(\sum_k \mathbb{Q}_{0k}\lambda_p^{\delta(k)}z^k),$$
$$\mathbb{S}^{\mathbb{P}^1}(H)|_p = e^{\frac{\mathbb{U}_p^{\mathbb{P}^1}}{z}}\lambda_p(\sum_k \mathbb{Q}_{1k}\lambda_p^{\delta(k)}z^k)$$

where $\delta(k) = 0$ if k is even, $\delta(k) = 1$ if k is odd. Here $\mathbb{Q}_{ik}$ are constant with respect to λ_p.

Remark 18. We can calculate $\mathbb{Q}_{ik}$ in closed form using Picard-Fuchs equation of I-function. Since this is not needed in our paper, we leave the details to the readers.

Proposition 19. *Let $\Gamma \in \mathsf{G}_{g,0}^{\mathrm{ord}}$ be a decorated graph.*

1. *$Cont_\Gamma^{\mathsf{A}}(v)$ has type $(\mathsf{g}(v)-1, 3\mathsf{g}(v)-3+N_v(\mathsf{A}))$, where $N_v(\mathsf{A}) = \sum_{i=1}^k(2-a_i)$. Here $(a_1,\dots,a_k)$ are components of A associated to v.*
2. *$Cont_\Gamma$ has type $(g-1, 3g-3)$.*

Proof. For the first parts, if we expand

$$\mathrm{Cont}_\Gamma^{\mathsf{A}}(v) = [\psi^{a_1-1},\psi^{a_2-1},...,\psi^{a_k-1}|\mathsf{H}_{\mathsf{g}(v)}^v]_{\mathsf{g}(v),k}^{0+,v}$$
$$= \sum_{l_1,l_2,...,l_{\mathsf{g}(v)}\geq 0} b_{l_1,l_2,...,l_{\mathsf{g}(v)}}[\psi^{a_1-1},\psi^{a_2-1},...,\psi^{a_k-1}|c_1^{l_1}c_2^{l_2}...c_{\mathsf{g}(v)}^{l_{\mathsf{g}(v)}}]_{\mathsf{g}(v),k}^{0+,v},$$

where $a_1, a_2, ..., a_k$ are components of A associated to v. Now we can check

- $b_{l_1,l_2,...,l_{\mathsf{g}(v)}}$ has type $(\mathsf{g}(v)-1, \sum_{i=1}^{\mathsf{g}(v)} l_i)$,
- $[\psi^{a_1-1}, ..., \psi^{a_k-1}|c_1^{l_1}c_2^{l_2}...c_g^{l_g}]^v_{\mathsf{g}(v),k}$ has type $(0, 3\mathsf{g}(v)_v - 3 - \sum_{i=1}^{\mathsf{g}(v)} l_i + \sum_{i=1}^{k}(2-a_i))$.

The proof of first part follows from Lemma 7.

For the second parts, let e be an edge connecting v_i and v_j. Then, by Prop 16 and (2), we can check that

$$\mathrm{Cont}^{\mathsf{A}}_{\Gamma}(e) = \left[(-1)^{k+l} e^{-\frac{\overline{\mathbb{U}}_i^{\mathbb{P}[n]}}{x} - \frac{\overline{\mathbb{U}}_j^{\mathbb{P}[n]}}{y}} e_i \overline{\mathbb{V}}_{ij}^{\mathbb{P}[n]}(x,y) e_j\right]_{x^{a_i-1}y^{a_j-1}}$$

$$= \left[(-1)^{k+l} e^{-\frac{\overline{\mathbb{U}}_i^{\mathbb{P}[n]}}{x} - \frac{\overline{\mathbb{U}}_j^{\mathbb{P}[n]}}{y}} \sum_k \overline{\mathbb{S}}^{\mathbb{P}[n]}(\phi_k)|_{v_i, z=x}\, \overline{\mathbb{S}}^{\mathbb{P}[n]}(\phi^k)|_{v_j,z=y}\right]_{x^k y^{l-1} - x^{k+1}y^{l-2} + ... + (-1)^{k-1}x^{k+l-1}}$$

has type $(1, 1+(a_i-1)+(a_j-1))$. Here (a_i, a_j) are components of A associated to flags (v_i, e) and (v_j, e). By multiplying all contributions of Γ, the proof of second part follows from Lemma 7.

□

4.2. Proof of Theorem 3

Consider the following decomposition of F^g,

$$F^g|_{\lambda_i=1, \overline{\lambda}_i=-1} = \sum_{\lambda_{i,j}=\pm 1} \sum_{\Gamma \in \mathsf{G}^{\mathrm{ord}}_{g,0}} \mathrm{Cont}_\Gamma.$$

Each Cont_Γ has type $(g-1, 3g-3)$ by Prop 19. Using Lemma 8, we obtain the following result for $n > 3g-3$ and even g,

$$\sum_{\lambda_{i,j}=\pm 1} \mathrm{Cont}_\Gamma = 0.$$

Therefore we conclude

$$F^g|_{\lambda_i=1, \overline{\lambda}_i=-1} = \sum_{\lambda_{i,j}=\pm 1} \sum_{\Gamma \in \mathsf{G}^{\mathrm{ord}}_{g,0}} \mathrm{Cont}_\Gamma = 0,$$

for $n > 3g-3$ and even g. The q-coefficients in F^g of degree d with virtual dimension 0 are independant of λ. Theroem 3 is immediate consequence.

5. Gromov-Witten invariants with insertions

Here we extend the main result to the Gromov-Witten invariants with insertions. Let π be morphism to the moduli space of stable curves determined by the domain,

$$\pi : Q_{g,n}(\mathbb{P}[n], \mathsf{d}) \to \overline{M}_{g,n}.$$

For $\mathsf{c}_k = (c_{1,k}, c_{2,k}, \dots, c_{m,k}) \in \mathbb{Z}_2^m$, Gromov-Witten invariants is defined by

$$GW^{\mathbb{P}[n]}_{g,m,\mathsf{d}}[\tau_{k_1}\mathsf{c}_1, \dots, \tau_{k_m}\mathsf{c}_m] := \int_{[\overline{M}_{g,m}(\mathbb{P}[n],\mathsf{d})]^{vir}} \prod_{i=1}^{m} \pi^*(\psi_i)^{k_i} \mathrm{ev}_i^*(\otimes_{j=1}^n H_j^{c_{j,i}})\,.$$

Define the equivariant quasimap series

$$F^g_m[\tau_{k_1}\mathsf{c}_1, \dots, \tau_{k_m}\mathsf{c}_m] \quad := \sum_{\mathsf{d}=0}^{\infty} \mathsf{q}^{\mathsf{d}} \int_{[Q_{g,m}(\mathbb{P}[n],\mathsf{d})]^{vir}} \prod_{i=1}^{m} \pi^*(\psi_i)^{k_i} \mathrm{ev}_i^*(\otimes_{j=1}^n H_j^{c_{j,i}})\,.$$

The methods of proof of Theorem 3 immediately yield the extended results for the equivariant quasimap series with insertions.

Theorem 20. *We have*

$$F^g_m[\tau_{k_1}\mathsf{c}_1, \dots, \tau_{k_m}\mathsf{c}_m]|_{\lambda_i=1,\overline{\lambda}_i=-1} = 0,$$

if the following conditions hold.

(i) $(g-1+\sum_{k=1}^m c_{ik})$ *is odd for* $1 \le i \le n$,
(ii) $3g-3+m < n$.

As a consequence, we can extend the Theorem 1 to the following result.

Theorem 21. *We have*

$$GW^{\mathbb{P}[n]}_{g,m,\mathsf{d}}[\tau_{k_1}\mathsf{c}_1, \dots, \tau_{k_m}\mathsf{c}_m] = 0,$$

if the following conditions hold.

(i) $(g-1+\sum_{k=1}^m c_{ik})$ *is odd for* $1 \le i \le n$,
(ii) $3g-3+m < n$.

References

[1] I. Ciocan-Fontanine and B. Kim, *Moduli stacks of stable toric quasimaps,* Adv. in Math., **225** (2010), 3022-3051.

[2] I. Ciocan-Fontanine and B. Kim, *Wall-crossing in genus zero quasimap theory and mirror maps,* Algebr. Geom., **1** (2014), 400-448.

[3] I. Ciocan-Fontanine and B. Kim, *Higher genus quasimap wall-crossing for semi-positive targets,* JEMS., **19** (2017), 2051-2102.

[4] I. Ciocan-Fontanine and B. Kim, *Big I-functions* in *Development of moduli theory Kyoto 2013*, Adv. Stud. Pure Math. **69**, Math. Soc. Japan, (2016), 323-347.

[5] I. Ciocan-Fontanine, B. Kim, and D. Maulik, *Stable quasimaps to GIT quotients,* J. Geom. Phys., **75** (2014), 17-47.

[6] D. Cox and S. Katz, *Mirror symmetry and algebraic geometry*, Mathematical Surveys and Monographs **68**: Amer. Math. Soc., Providence, RI, 1999.

[7] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, Algebraic geometry – Santa Cruz 1995, 45-96, Proc. Sympos. Pure Math. **62**, Part 2: Amer. Math. Soc., Providence, RI, 1997.

[8] A. Givental, *Equivariant Gromov-Witten invariants*, Internat. Math. Res. Notices., **13** (1996), 613-663.

[9] A. Givental, *Elliptic Gromov-Witten invariants and the generalized mirror conjecture,* Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), 107-155, World Sci. Publ., River Edge, NJ, 1998.

[10] A. Givental, *Semisimple Frobenius structures at higher genus*, Internat. Math. Res. Notices., **23** (2001), 613-663.

[11] T. Graber, R. Pandharipande, *Localization of virtual classes*, Invent. Math., **135** (1999), 487-518.

[12] B. Kim and H. Lho, *Mirror theorem for elliptic quasimap invariants*, Geom. Topol. **22** (2018), 1459-1481.

[13] Y.-P. Lee and R. Pandharipande, *Frobenius manifolds, Gromov-Witten theory and Virasoro constraints,* https://people.math.ethz.ch/˜rahul/, 2004.

[14] H. Lho and R. Pandharipande, *Stable quotients and holomorphic anomaly equation*, Adv. Math., **332** (2018), 349-402.

[15] A. Marian, D. Oprea, Dragos, R. Pandharipande, *The moduli space of stable quotients,* Geom. Topol., **15** (2011), 1651-1706.

Department of Mathematics
Chungnam National University
Daejon 34134, Republic of Korea
E-mail: hyenho@cnu.ac.kr